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Redshift in Hubble's Constant

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A topological field theory with Bogomol'nyi solitons is examined. The Bogomol'nyi solitons have much in common with the instanton in Yang–Mills theory; consequently we called them 'topological instantons.' When periodic boundary conditions are imposed, the field theory comments indirectly on the speed of light within the theory. In this particular model the speed of light is not a universal constant. This may or may not be relevant to the current debate in astronomy and cosmology over the large values of the Hubble constant obtained by the latest generation of ground- and space-based telescopes. An experiment is proposed to detect spatial variation in the speed of light.

1. INTRODUCTION

Astronomers have recently reported on observations for the Hubble constant that predict an age for the universe younger than the estimated age of some nearby globular clusters (Pierce *et al.*, 1994; Freedman *et al.*, 1994; Tanvir *et al.*, 1995). The obvious tension has been christened the 'age crisis.' There are three possibilities, of course: the measurements made for the Hubble constant are incorrect; current models for stellar evolution are incorrect; or there is new physics to be understood. It is unanimously agreed that without further observation it is too early to judge. And, as noted by Sandage (1993), it must be understood why other observations for the Hubble constant based on type Ia supernovae are only half as large.

Observation has uncovered other pecularities in Hubble's constant. The Hubble relation turns sharply upward from linearity on a redshift vs. distance plot (Kristian *et al.*, 1978; Spinrad *et al.*, 1987). Refinements to the distance scale cannot of course account for the nonlinearity. Sandage argues that the

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nonlinearity is due to a bias in the choice of objects chosen for observation—we choose those objects that we are capable of seeing. This bias must be filtered out in order to determine correctly the Hubble constant.

Another hypothesis, although seemingly a remote possibility, is suggested by the observations: the speed of light in vacuum at distant space-time points from earth is less than the present, terrestrial speed of light. For suppose that the speed of light in a vacuum were not an absolute constant; then the redshift would need to be reassessed. The redshift is given by

$$\frac{\Delta\lambda}{\lambda} \simeq \frac{v}{\tilde{c}} \tag{1}$$

where we assume that the velocity of the emitter, v, is much less than the speed of light when the photon is emitted, $\tilde{c}$. If the speed of light at the time of emission is smaller than the present speed of light on earth, then the observed redshift in (1) would be greater. This would make the Hubble constant larger, and thereby make the universe appear to be younger than it actually is. A lower speed of light in the early universe might thereby mitigate the 'age crisis.' In addition, one notes that under these circumstances the enormous powers associated with quasars would be reduced correspondingly.

It is commonly held that a theoretical derivation for the speed of a light will require a theory of quantum gravity. A theory of quantum gravity is defined to be any theory that treats the space-time metric as a quantum field, so that quantum fluctuations in the light cone are expected. At some point, however, the light cone must be fixed and the speed of light determined. Since a quantum gravity theory does not exist at present, this cannot be investigated further. In the next section we introduce a generally covariant, classical, gauge field theory containing Yang–Mills and electrodynamics, which interestingly comments on the speed of light within that particular theory. The speed of light is found to vary spatially. A model therefore exists that lends a degree of support to the hypothesis above that the speed of light is not a universal constant. In Section 3 we examine how one might detect spatial variation in the speed of light. Note that our calculations lie entirely with classical field theory, and that although this paper studies specifically the model introduced in the next section, the approach and philosophy operate in a much larger context.

2. A MODEL WITH A VARIABLE SPEED OF LIGHT

Let π: $P \to \mathbf{R}^4$ be a principal $U(2)$-bundle over $\mathbf{R}^4$, and denote by E the associated rank-two adjoint vector bundle. $\mathcal{A}(P)$ is the space of connections on P. Let A, $B \in \mathcal{A}(P)$, and introduce local coordinate charts with indices $\mu = 0, \ldots, 3$ on $\mathbf{R}^4$. The Lie-algebra-valued connections or vector potentials A_μ

and B_μ induce exterior covariant derivatives $D^A_\mu = \partial_\mu + A_\mu$ and $D^B_\mu = \partial_\mu + B_\mu$ on the associated adjoint vector bundle E. The curvatures H^A and K^B are defined by $2D^A_{[\mu}D^A_{\nu]}s = H^A_{\mu\nu}s$ and $2D^B_{[\mu}D^B_{\nu]}s = K^B_{\mu\nu}s$, respectively. In this way H^A and K^B are two-forms on $\mathbf{R}^4$ taking values in the adjoint bundle E. In addition, we introduce two Higgs fields: Φ_A and Φ_B are functions on $\mathbf{R}^4$ taking values in the adjoint bundle E. The Lagrangian action that forms the basis of our model is given by

$$\mathcal{L}(A, B, \Phi_A, \Phi_B) = \int_{\mathbf{R}^4} \langle (H^A \otimes \Phi_B) \wedge (\Phi_A \otimes K^B) \rangle - \tfrac{1}{2} \langle (\Phi_A \otimes K^B)^2 \rangle \tag{2}$$

The wedge product is taken on the space-time indices, and the tensor product is on the Lie-algebra vector spaces. The bundle inner product—the Hermitian structure—is denoted by $\langle \cdot \rangle$. The Killing–Cartan inner product can be adopted. In any case, the Hermitian structure is normalized so that $\langle I_E^2 \rangle = 1$. To make the action finite, we shall assume that the gauge fields are periodic in both space and time. Restricting consideration to one period, we compactify the underlying space-time manifold by adding the boundary and identifying. We require that the action be finite only over one period. The compactified space-time is now a torus T of real dimension four.

The form of the Lagrangian (2) generalizes the topological gauge field theories studied by Horowitz (1989). In local space-time coordinates and using the Killing–Cartan inner product, we can write the Lagrangian action explicitly as

$$\begin{aligned}\mathcal{L}(A, B, \Phi_A, \Phi_B) &= \int_T H^a_{[\mu\nu} K^b_{\lambda\rho]} \operatorname{tr}(T^a \Phi_A) \operatorname{tr}(T^b \Phi_B)\, d^4x \\ &\quad - \frac{1}{2} \int_T K^a_{[\mu\nu} K^b_{\lambda\rho]} \operatorname{tr}(\Phi_A \Phi_B) \operatorname{tr}(T^a T^b)\, d^4x \end{aligned} \tag{3}$$

The generators of the Lie algebra are denoted by T^a. For gauge groups where $\operatorname{tr}(T^a) = 0$, the Lagrangian (3) reduces to the second integral—these are the topological field theories studied by Baulieu and Singer (1988). The variational field equations for the Lagrangian (2) and for arbitrary Φ_A and Φ_B are

$$D^A K^B = 0, \qquad D^B H^A = 0 \tag{4}$$

The field equations are clearly independent of any metric structure. The set of solutions to (4) is not trivial, because when $A = B$ the field equations reduce to the Bianchi identities.

Observe that when a space-time metric is placed on $\mathbf{R}^4$ (by whatever means) and used to define the Hodge star operator *, the topological field equations (4) become the source-free Yang–Mills or electrodynamic field equations when the gauge field B is chosen so that $K^B = *H^A$. The topological field theory above therefore is seen to contain Yang–Mills theory and electromagnetism. It is for this reason that we have defined a field theory with two vector potentials instead of one.

We turn now to the Bogomol'nyi structure. The Higgs fields will be arbitrary no longer. Set both Φ_A and Φ_B equal to I_E, the identity endomorphism on the adjoint bundle E. By completing the square, we can rewrite the Lagrangian (2) as

$$2\mathcal{L} = \int_T \langle (H^A \otimes I_E - I_E \otimes K^B)^2 \rangle - \int_T \langle (H^A \otimes I_E)^2 \rangle \tag{5}$$

The Lagrangian (5) is now in Bogomol'nyi form with Bogomol'nyi equations

$$H^A \otimes I_E - I_E \otimes K^B = 0 \tag{6}$$

By a computation on the indices, equations (6) imply that H^A and K^B are projectively flat,

$$H^A = K^B = iFI \tag{7}$$

where F is a real-valued two-form on T. Clearly solutions to the Bogomol'nyi equations (6) automatically satisfy the variational field equations (4) when F is closed.

Let E_A and E_B be the adjoint vector bundles equipped with the covariant derivatives D^A and D^B, respectively. E^* is the dual bundle to E. Recall that the curvature of the tensor product bundle $E_A \otimes E_B^*$ is given by (Kobayashi, 1987)

$$\Omega_{E_A \otimes E_B^*} = H^A \otimes I_E - I_E \otimes K^B$$

The Bogomol'nyi equations in (6) are now seen to be a vanishing curvature condition on the tensor product bundle $E_A \otimes E_B^*$. The trace inner product on E generalizes naturally to an inner product on $E \otimes E^*$. The first term in (5) is proportional to the topological characteristic class:

$$c_2(E \otimes E^*) - \tfrac{1}{2}c_1(E \otimes E^*)^2 = 2nc_2(E) - (n - 1)c_1(E)^2 \tag{8}$$

In order that the vector bundle E admit a projectively flat connection, the characteristic class (8) necessarily vanishes. The second integral in (5) is also a characteristic class,

$$\int_T \mathrm{tr}(H^A \wedge H^A) = 8\pi^2\left(c_2(E) - \frac{1}{2}c_1(E)^2\right) = -8\pi^2\chi(E) \tag{9}$$

The Euler characteristic of the bundle E is denoted by $\chi(E)$. It is clear that under a perturbation of the vector potential both integrals in (5) are invariant, and when the Bogomol'nyi equations are satisfied the Lagrangian is proportional to the Euler characteristic (9). Nonsingular solutions to the Bogomol'nyi equations are nontrivial and stable when the Euler characteristic is nonvanishing.

Of particular interest to us is the moduli space of solutions to the Bogomol'nyi equations, because the moduli space is the (covariant) phase space. In algebraic geometry, mathematicians impose Mumford–Takemoto topological stability on gauge theories to ensure that the moduli spaces are topologically well-behaved. Kobayashi (1987) has reformulated Mumford–Takemoto stability into a differential geometric form, known as the Einstein–Hermitian condition. The Einstein–Hermitian condition is well suited to the problem we are investigating, as we shall see, because the Einstein–Hermitian condition is closely related to the projective flatness produced by our Bogomol'nyi structure.

To implement the Einstein–Hermitian condition we shall assume that after imposing periodic boundary conditions, the resulting four-torus is equipped with a Kähler structure, and $E \to T$ is equipped with a holomorphic structure. All complex tori admit a Kähler structure: a Kähler metric g and a closed Kähler form Φ of type (1, 1). The Hermitian metric $\langle \cdot \rangle$ and a holomorphic structure $\bar{\partial}$ on E give rise to a unique connection A. We shall now free the Hermitian metric and allow it to vary up to a conformal transformation. The curvature H^A is of type (1, 1). The mean curvature K of the vector bundle over a Kähler manifold is Einstein–Hermitian when the mean curvature K satisfies (Kobayashi, 1987, p. 99)

$$K = kI_E \tag{10}$$

where k is a real constant. Compare equation (10) with the projective flatness condition,

$$H^A = FI_E \tag{11}$$

of the Bogomol'nyi equations (7). It has been proved in the mathematical literature that up to a conformal change in the Hermitian structure, a holomorphic projectively flat connection and an Einstein–Hermitian connection on a vector bundle satisfying (8) are equivalent (Kobayashi, 1987; Lübke, 1982). The value of the constant k is given by

$$k = 2\pi \int_M c_1(E) \wedge \Phi \Big/ \int_M \Phi^2 = 2\pi \frac{\deg(E)}{\mathrm{vol}(M)}$$

To obtain a phase space that is topologically well-behaved we shall therefore restrict ourselves to the set of holomorphic projectively flat connections on a vector bundle E satisfying the topological condition (8), equivalently, Einstein–Hermitian connections. We call holomorphic Bogomol'nyi solitons 'topological instantons.' For fixed k, the phase space is denoted by $\mathcal{M}_k$.

The complex rank-two vector bundle $(E, \langle\cdot\rangle) \rightarrow (T, g, \Phi)$ defined over the Kähler torus is assumed to have the Chern numbers $4c_2(E) = c_1(E)^2 = -4$. Our choice of topology is such that the topological condition (8) is satisfied and the Lagrangian (9) is nonvanishing. This is sufficient for the bundle to admit stable topological instantons, although existence remains at issue. We shall study 'diagonal' $U(2)$ topological instantons on this bundle. By 'diagonal' we mean that the Einstein–Hermitian connections A and B are equal ($A = B$). Diagonal instantons are examined because under normal circumstances there is little physical evidence to suggest two distinct vector potentials are necessary. We take the constant k in the Einstein–Hermitian condition (10) to be fixed and nonzero. The Kähler structure on T allows us to say a great deal about the phase space $\mathcal{M}_k$.

The complex dimension of the $U(2)$ topological instanton phase space when nonempty is given by (Kobayashi, 1987)

$$\dim_{\mathbf{C}}(\mathcal{M}_k) = 4h^{0,1}(M) - 6 \tag{12}$$

The Kähler torus has $h^{0,1}(M) = 2$. Therefore if a (diagonal) topological instanton exists, the real dimension of the phase space $\mathcal{M}_k$ is four. It is significant that $\dim(\mathcal{M}_k) = 4$, because it is only massless particles that have a phase space of real dimension four in (3 + 1) space-time. [We have neglected $K3$ surfaces as models for compactified space-time because they have $h^{0,1}(K3) = 0$, thereby giving the phase space negative dimension.]

Our next task is to include special relativity into the theory by locally mapping the phase space $\mathcal{M}_k$ to the model phase space for massless particles. Consider the phase space for massless particles in $\mathbf{R}^3$. The (covariant) phase space is equivalent to the space-of-motions. Massless particles in $\mathbf{R}^3$ move on straight lines and at the speed of light c. We may therefore parametrize the possible motions of a massless particle by assigning to a straight line $\mathbf{x}(t)$ in $\mathbf{R}^3$ a velocity vector $\mathbf{c} = c\hat{\mathbf{n}}$ and a position vector $\mathbf{d}$ so that $\mathbf{x}(t) = \mathbf{d} + t\mathbf{c}$. The position vector $\mathbf{d}$ is defined as the normal from the origin to the line $\mathbf{x}(t)$, equivalently, the point on the line nearest the origin. See Fig. 1. Thus the phase space is

$$\mathcal{M}_{\text{massless}} \cong \{(\mathbf{c}, \mathbf{d}) \in S_c^2 \times R^3 \mid \mathbf{c}\cdot\mathbf{d} = 0\} \tag{13}$$

Therefore the model phase space for a massless particle on $\mathbf{R}^3$ is equivalent to the tangent bundle TS_c^2 where the radius of the sphere is the speed of light. The natural metric on TS_c^2 is given by

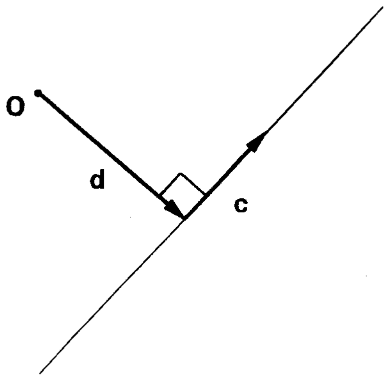

Fig. 1. A physical parametrization of the space-of-motions for massless particles in $\mathbf{R}^3$.

$$ds^2 = f(r)\,dr^2 + a(r)(d\psi + \cos\theta\,d\phi)^2 + c^2(d\theta^2 + \sin^2\theta\,d\phi^2) \quad (14)$$

where (θ, ϕ) are spherical polar coordinates on the two-sphere and $(r = \|\mathbf{d}\|, \psi)$ plane polar coordinates on the tangent plane. The functions $f(r)$ and $a(r)$ are arbitrary, and we have multiplied c by unit time so that both $\mathbf{d}$ and c have units of length. We shall require that the local geometry of $\mathcal{M}_k$ reduce to the geometry of special relativity.

Using Fourier–Mukai transforms, we find that some theorems are available. Assume that the real four-dimensional space-time torus is an Abelian variety with a $(1, 2r)$ polarization l. It has been proved that the phase space $\mathcal{M}_k$ is nonempty (that is, topological instantons exist) and isomorphic to the four-torus T (Maciocia, 1995, Proposition 7.1). By the isomorphism $T \simeq \mathcal{M}_k$, the phase space is equipped with a natural Kähler metric with vanishing Ricci tensor. In fact it is the flat metric. The model metric (14) implies that the speed of light is infinite. We have been hasty, however. Recall that points were added to space-time in order to compactify it. We must now remove those points. The isomorphism $T \simeq \mathcal{M}_k$ allows us to remove the corresponding points in the phase space. The phase space is now decompactified. The metric on the phase space is no longer necessarily flat. Since $\mathcal{M}_k$ remains four-dimensional and still has vanishing Ricci tensor, the phase space is hyper-Kähler. Assume that the hyperKähler metric is complete and nonsingular on the decompactified topological instanton phase space $\tilde{\mathcal{M}}_k$. Assume local isotropy of the universe, so that the phase space $\tilde{\mathcal{M}}_k$ admits $SO(3)$ as a group of local isometries. Let us also assume that on $\tilde{\mathcal{M}}_k$ the orbits defined by the action under the isometries are generically three-dimensional. Then, the only complete, nonsingular, $SO(3)$-invariant hyperKähler metrics on four-manifolds with three-dimensional orbits are: the flat metric, the Atiyah–Hitchin metric, the Taub–NUT metric with positive mass, and the Eguchi–Hanson metric (Gibbons and Ruback, 1988). The Taub–NUT and Eguchi–Hanson metrics admit another $U(1)$ to make them $U(2)$-invariant. Only the Taub–NUT and the Eguchi–Hanson metrics, appear to be remotely compatible with the massless particle metric (14). If we also place on the model phase space M_{massless} its natural complex structure and note that it is invariant under the natural $SO(3)$ action, then only the Eguchi–Hanson metric is compatible with the local geometry of special relativity.

The Eguchi–Hanson metric is of the form

$$ds^2 = [\gamma(r)]^{-1}\, dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + \gamma(r)\sigma_3^2)$$

where $\gamma(r) \equiv 1 - (a/r)^4$ and $\{\sigma_i\}$ is the dual basis for *so* (3). In terms of Euler angles, we define

$$\sigma_1 = d\phi \sin\theta \cos\psi - d\theta \sin\psi$$

$$\sigma_2 = d\phi \sin\theta \sin\psi + d\theta \cos\psi$$

$$\sigma_3 = d\phi \cos\theta + d\psi$$

In Euler coordinates the Eguchi–Hanson metric becomes

$$ds^2 = [\gamma(r)]^{-1}\, dr^2 + \frac{r^2}{4}\gamma(r)(d\psi + \cos\theta\, d\phi)^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{15}$$

Compare (15) with (14). Note that while the metrics are certainly similar, the two-sphere in the Eguchi–Hanson metric is a function of r. With this observation our physical parametrization follows.

The Euler angles (θ, ϕ, ψ) in the Eguchi–Hanson metric (15) define the direction of the propagation, and the radius of the sphere is the speed of the massless topological instanton, as we saw for the model massless particle above. For the instanton to carry both energy and momentum and still remain massless, the energy-momentum relation in special relativity implies that the topological instanton moves at the speed of light. Thus the speed of light in this theory is also subject to spatial variation. However, the spatial coordinate r in the tangent space is ambiguous. Where, for example, is the origin? In conjunction with the cosmological principle we could place with impunity the origin on earth. The debate over the Hubble constant suggests that as a first-order approximation we try $r = 2c - \alpha\|\mathbf{d}\|$, where c is the present terrestrial speed of light in vacuum multiplied by unit time, α is a dimensionless constant, and $\|\mathbf{d}\|$ is the distance from earth. The constant α is then determined through observation and experiment. If the cosmological principle were not to hold, then the placement of the origin(s) is very important. One must, of course, rely on observation to place it/them. A final note: the quantization of the field theory may also have something significant to add to the physical parametrization of the phase space. We shall examine this in a forthcoming paper.

3. COMPTON SCATTERING

Assume that the speed of light $\tilde{c}$ is not an absolute constant when viewed at very large spatial scales. To measure deviations in speed between distant light $\tilde{c}$ and terrestrial light c, one presumably examines photons that have interacted with matter in the early universe and have since then traveled unimpeded through space. Some of these photons eventually enter a detector; unimpeded travel requires that the detector be space-based. Since it is assumed that no interaction occurs during the photons' long journey, the energy E and the linear momentum $E/\tilde{c}$ are conserved. This implies that the photons travel toward earth with the constant speed of light $\tilde{c}$ given to them upon emission. The incoming photons are absorbed by a loosely bound electron assumed to be initially at rest with mass m and are reemitted (photon scattering). We shall assume that the energy and linear momentum are conserved in photon scattering. By studying the scattered photons we determine characteristics of the incoming photons. This is the Compton effect, of course. The standard computation using conservation of momentum and energy gives the lowest order correction to the Compton formula. Let θ denote the scattering angle, and define $c = \tilde{c} + \Delta c$, $\epsilon \equiv \Delta c/\tilde{c}$, and $\Delta\lambda = \lambda' - \lambda$; then

$$\Delta\lambda \cong \frac{h}{mc}(1 - \cos\theta) + \epsilon\left[\lambda + \frac{1}{\lambda}\frac{h^2}{m^2c^2}(1 - \cos\theta)\right]$$

The second term is dependent on the wavelength, while the terrestrial Compton effect is obviously independent of the wavelength. Scattering dependence on the incident wavelength would be a clear signal for spatial variation in the speed of light.

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